Paper Specific Instructions

1. The examination is of 3 hours duration. There are a total of 60 questions carrying 100 marks. The entire paper is divided into three sections, **A**, **B** and **C**. All sections are compulsory. Questions in each section are of different types.

2. **Section – A** contains a total of 30 **Multiple Choice Questions (MCQ).** Each MCQ type question has four choices out of which only **one** choice is the correct answer. Questions Q.1 – Q.30 belong to this section and carry a total of 50 marks. Q.1 – Q.10 carry 1 mark each and Questions Q.11 – Q.30 carry 2 marks each.

3. **Section – B** contains a total of 10 **Multiple Select Questions (MSQ).** Each MSQ type question is similar to MCQ but with a difference that there may be **one or more than one** choice(s) that are correct out of the four given choices. The candidate gets full credit if he/she selects all the correct answers only and no wrong answers. Questions Q.31 – Q.40 belong to this section and carry 2 marks each with a total of 20 marks.

4. **Section – C** contains a total of 20 **Numerical Answer Type (NAT)** questions. For these NAT type questions, the answer is a real number which needs to be entered using the virtual keyboard on the monitor. No choices will be shown for these type of questions. Questions Q.41 – Q.60 belong to this section and carry a total of 30 marks. Q.41 – Q.50 carry 1 mark each and Questions Q.51 – Q.60 carry 2 marks each.

5. In all sections, questions not attempted will result in zero mark. In **Section – A** (MCQ), wrong answer will result in **NEGATIVE** marks. For all 1 mark questions, 1/3 marks will be deducted for each wrong answer. For all 2 marks questions, 2/3 marks will be deducted for each wrong answer. In **Section – B** (MSQ), there is **NO NEGATIVE** and **NO PARTIAL** marking provisions. There is **NO NEGATIVE** marking in **Section – C** (NAT) as well.

6. Only Virtual Scientific Calculator is allowed. Charts, graph sheets, tables, cellular phone or other electronic gadgets are **NOT** allowed in the examination hall.

7. The Scribble Pad will be provided for rough work.

Special Instructions/ Useful Data

$\mathbb{R}$	The set of real numbers
$\mathbb{R}^n$	$\{(x_1, x_2, \dots, x_n) : x_i \in \mathbb{R}, i = 1, 2, \dots, n\}$
$\det(M)$	Determinant of a matrix M
I_n	Identity matrix of order $n \times n, n = 2, 3, \dots$
g'	First derivative of a real valued function g
g''	Second derivative of a real valued function g
F^c	Complement of an event F
$P(F)$	Probability of an event F
$P(F\|G)$	Conditional probability of an event F given the occurrence of event G
$X \sim f$	The probability density/mass function of the random variable X is f
$E(X)$	Expectation of a random variable X
$\mathrm{Var}(X)$	Variance of a random variable X
$U(a, b)$	Continuous uniform distribution on the interval $(a, b), -\infty < a < b < \infty$
$\mathrm{Poisson}(\theta)$	Poisson distribution with mean $\theta, \theta \in (0, \infty)$
$N(\mu, \sigma^2)$	Normal distribution with mean μ and variance σ^2, $\mu \in (-\infty, \infty), \sigma^2 \in (0, \infty)$
χ_n^2	Central chi-square distribution with n degrees of freedom, $n = 1,2, \dots$
$F_{m,n}$	F distribution with (m, n) degrees of freedom, $m, n = 1,2, \dots$
$\Phi(\cdot)$	Distribution function of $N(0, 1)$
$\|x\|$	Absolute value of x
MLE	Maximum Likelihood Estimator
$n!$	$n \cdot (n-1) \cdots 3 \cdot 2 \cdot 1,\ n = 1, 2, 3, \dots,$ and $0! = 1$
$\binom{n}{k}$	$\frac{n!}{k!\,(n-k)!}, k = 0,1,2, \dots, n$ and $n = 1, 2, \dots;\ \binom{0}{0} = 1$
$\max\{a_1, a_2, \dots, a_n\}$	Maximum of real numbers $a_1, a_2, \dots, a_n$ $(n \geq 2)$
$\min\{a_1, a_2, \dots, a_n\}$	Minimum of real numbers $a_1, a_2, \dots, a_n$ $(n \geq 2)$
$\ln x$	Natural logarithm of x

SECTION – A

MULTIPLE CHOICE QUESTIONS (MCQ)

Q. 1 – Q.10 carry one mark each.

Q.1 The value of the limit

$$\lim_{n\to\infty}\left(\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\cdots\left(1+\frac{n}{n}\right)\right)^{\frac{1}{n}}$$

is equal to

(A) e (B) $\frac{1}{e}$ (C) $\frac{3}{e}$ (D) $\frac{4}{e}$

Q.2 Let $f:\mathbb{R}\to\mathbb{R}$ be a function defined by

$$f(x) = x^7 + 5\,x^3 + 11\,x + 15,\quad x\in\mathbb{R}.$$

Then, which of the following statements is **TRUE**?

(A) f is both one-one and onto

(B) f is neither one-one nor onto

(C) f is one-one but NOT onto

(D) f is onto but NOT one-one

Q.3 The value of the limit

$$\lim_{x\to 0}\frac{e^{-3x}-e^{x}+4x}{5(1-\cos x)}$$

is equal to

(A) 1 (B) 0 (C) $\frac{2}{5}$ (D) $\frac{8}{5}$

Q.4 The value of the limit

$$\lim_{n\to\infty} \sum_{k=0}^{n} \binom{2n}{k} \frac{1}{4^n}$$

is equal to

(A) 1 (B) $\frac{1}{2}$ (C) 0 (D) $\frac{1}{4}$

Q.5 Let $\{X_n\}_{n\geq 1}$ be a sequence of independent and identically distributed random variables with probability density function

$$f(x) = \begin{cases} 1, & \text{if } 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}.$$

Then, the value of the limit

$$\lim_{n\to\infty} P\left(-\frac{1}{n}\sum_{i=1}^{n} \ln X_i \leq 1 + \frac{1}{\sqrt{n}}\right)$$

is equal to

(A) $\frac{1}{2}$ (B) $\Phi(1)$ (C) 0 (D) $\Phi(2)$

Q.6 Let X be a $U(0,1)$ random variable and let $Y = X^2$. If ρ is the correlation coefficient between the random variables X and Y, then $48\,\rho^2$ is equal to

(A) 48 (B) 45 (C) 35 (D) 30

Q.7 Let M be a 3×3 real matrix. Let $\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}$ and $\begin{pmatrix}0\\-1\\\alpha\end{pmatrix}$ be the eigenvectors of M corresponding to three distinct eigenvalues of M, where α is a real number. Then, which of the following is **NOT** a possible value of α?

(A) 0 (B) 1 (C) -2 (D) 2

Q.8 If the series $\sum_{n=1}^{\infty} a_n$ converges absolutely, then which of the following series diverges?

(A) $\sum_{n=1}^{\infty} |a_{2n}|$

(B) $\sum_{n=1}^{\infty} \frac{a_n + a_{n+1}}{2}$

(C) $\sum_{n=1}^{\infty} (a_n)^3$

(D) $\sum_{n=2}^{\infty} \left(\frac{1}{(\ln n)^2} + a_n\right)$

Q.9 There are three urns, labeled, Urn 1, Urn 2 and Urn 3. Urn 1 contains 2 white balls and 2 black balls, Urn 2 contains 1 white ball and 3 black balls and Urn 3 contains 3 white balls and 1 black ball. Consider two coins with probability of obtaining head in their single trials as 0.2 and 0.3. The two coins are tossed independently once, and an urn is selected according to the following scheme: Urn 1 is selected if 2 heads are obtained; Urn 3 is selected if 2 tails are obtained; otherwise Urn 2 is selected. A ball is then drawn at random from the selected urn. Then

$$P(\text{Urn 1 is selected} \mid \text{the ball drawn is white})$$

is equal to

(A) $\frac{6}{109}$

(B) $\frac{12}{109}$

(C) $\frac{1}{18}$

(D) $\frac{1}{9}$

Q.10 Let X be a random variable with probability density function

$$f(x) = \frac{1}{2} e^{-|x|}, \quad -\infty < x < \infty.$$

Then, which of the following statements is **FALSE**?

(A) $E(X\,|X|) = 0$

(B) $E(X\,|X|^2) = 0$

(C) $E\left(|X| \sin\left(\frac{X}{|X|}\right)\right) = 0$

(D) $E\left(|X| \sin^2\left(\frac{X}{|X|}\right)\right) = 0$

Q. 11 – Q. 30 carry two marks each.

Q.11 Let $f: \mathbb{R}^2 \to \mathbb{R}$ be a function defined by

$$f(x,y) = \begin{cases} \dfrac{y^3}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}.$$

Let $f_x(x,y)$ and $f_y(x,y)$ denote the first order partial derivatives of $f(x,y)$ with respect to x and y, respectively, at the point (x,y). Then, which of the following statements is **FALSE**?

(A) $f_x(x,y)$ exists and is bounded at every $(x,y) \in \mathbb{R}^2$

(B) $f_y(x,y)$ exists and is bounded at every $(x,y) \in \mathbb{R}^2$

(C) $f_y(0,0)$ exists and $f_y(x,y)$ is continuous at $(0,0)$

(D) f is NOT differentiable at $(0,0)$

Q.12 Let $\{X_n\}_{n\geq 1}$ be a sequence of independent and identically distributed $N(0,1)$ random variables. Then,

$$\lim_{n\to\infty} P\left(\frac{\sum_{i=1}^{n} X_i^4 - 3n}{\sqrt{32\,n}} \leq \sqrt{6}\right)$$

is equal to

(A) $\frac{1}{2}$ (B) $\Phi(\sqrt{2}\,)$ (C) 0 (D) $\Phi(1)$

Q.13 Consider a sequence of independent Bernoulli trials with probability of success in each trial as $\frac{1}{3}$.

The probability that three successes occur before four failures is equal to

(A) $\frac{179}{243}$ (B) $\frac{179}{841}$ (C) $\frac{233}{729}$ (D) $\frac{179}{1215}$

Q.14 Let X and Y be independent $N(0,1)$ random variables and $Z = \frac{|X|}{|Y|}$. Then, which of the following expectations is finite?

(A) $E\left(\frac{1}{\sqrt{Z}}\right)$ (B) $E\left(Z\sqrt{Z}\right)$ (C) $E(Z)$ (D) $E\left(\frac{1}{Z\sqrt{Z}}\right)$

Q.15 Consider three coins having probabilities of obtaining head in a single trial as $\frac{1}{4}, \frac{1}{2}$ and $\frac{3}{4}$, respectively. A player selects one of these three coins at random (each coin is equally likely to be selected). If the player tosses the selected coin five times independently, then the probability of obtaining two tails in five tosses is equal to

(A) $\frac{85}{384}$ (B) $\frac{255}{384}$

(C) $\frac{125}{384}$ (D) $\frac{64}{384}$

Q.16 Let X be a random variable having the probability density function

$$f(x) = \begin{cases} e^{-x}, & x > 0 \\ 0, & x \leq 0 \end{cases}.$$

Define $Y = [X]$, where $[X]$ denotes the largest integer not exceeding X. Then, $E(Y^2)$ is equal to

(A) $\frac{e(e+1)}{e-1}$ (B) $\frac{e+1}{(e-1)^2}$

(C) $\frac{e(e+1)^2}{e-1}$ (D) $\frac{(e+1)^2}{(e-1)^2}$

Q.17 Let X be a continuous random variable having the moment generating function

$$M(t) = \frac{e^t - 1}{t}, \qquad t \neq 0.$$

Let $\alpha = P(48\,X^2 - 40\,X + 3 > 0)$ and $\beta = P((\ln X)^2 + 2\,\ln X - 3 > 0)$.

Then, the value of $\alpha - 2\ln\beta$ is equal to

(A) $\frac{10}{3}$ (B) $\frac{19}{3}$ (C) $\frac{13}{3}$ (D) $\frac{17}{3}$

Q.18 Let $X_1, X_2, \ldots, X_n$ $(n \geq 3)$ be a random sample from Poisson(θ), where $\theta \in (0, \infty)$ is unknown and let

$$T = \sum_{i=1}^{n} X_i.$$

Then, the uniformly minimum variance unbiased estimator of $e^{-2\theta}\theta^3$

(A) is $\frac{T}{n}\left(\frac{T}{n} - 1\right)\left(\frac{T}{n} - 2\right)\left(1 - \frac{2}{n}\right)^{T-3}$

(B) is $\frac{T(T-1)(T-2)(n-2)^{T-3}}{n^T}$

(C) does NOT exist

(D) is $e^{-\frac{2T}{n}}\left(\frac{T}{n}\right)^3$

Q.19 Let $X_1, X_2, \ldots, X_n$ $(n \geq 2)$ be a random sample from $U(\theta - 5, \theta + 5)$, where $\theta \in (0, \infty)$ is unknown. Let $T = \max\{X_1, X_2, \ldots, X_n\}$ and $U = \min\{X_1, X_2, \ldots, X_n\}$. Then, which of the following statements is **TRUE**?

(A) $\frac{T+U}{2}$ is the unique MLE of θ

(B) $\frac{2}{T+U}$ is an MLE of $\frac{1}{\theta}$

(C) MLE of $\frac{1}{\theta}$ does NOT exist

(D) $U + 8$ is an MLE of θ

Q.20 Let X and Y be random variables having chi-square distributions with 6 and 3 degrees of freedom, respectively. Then, which of the following statements is **TRUE**?

(A) $P(X > 0.7) > P(Y > 0.7)$

(B) $P(X > 0.7) < P(Y > 0.7)$

(C) $P(X > 3) < P(Y > 3)$

(D) $P(X < 6) > P(Y < 6)$

Q.21 Let (X, Y) be a random vector with joint moment generating function

$$M(t_1, t_2) = \frac{1}{(1-(t_1+t_2))(1-t_2)}, \qquad -\infty < t_1 < \infty, -\infty < t_2 < \min\{1, 1-t_1\}.$$

Let $Z = X + Y$. Then, $\mathrm{Var}(Z)$ is equal to

(A) 3 (B) 4 (C) 5 (D) 6

Q.22 Let X be a continuous random variable with distribution function

$$F(x) = \begin{cases} 0, & \text{if } x < 0 \\ a\,x^2, & \text{if } 0 \le x < 2, \\ 1, & \text{if } x \ge 2 \end{cases}$$

for some real constant a. Then, $E(X)$ is equal to

(A) $\frac{4}{3}$ (B) $\frac{1}{4}$ (C) 1 (D) 0

Q.23 Let $X_1, X_2, \dots, X_n$ be a random sample from an exponential distribution with probability density function

$$f(x;\theta) = \begin{cases} \theta e^{-\theta x}, & x > 0 \\ 0, & \text{otherwise} \end{cases},$$

where $\theta \in (0,\infty)$ is unknown. Let $\alpha \in (0,1)$ be fixed and let β be the power of the most powerful test of size α for testing $H_0: \theta = 1$ against $H_1: \theta = 2$.

Consider the critical region

$$R = \left\{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : \sum_{i=1}^{n} x_i > \frac{1}{2}\chi^2_{2n}(1-\alpha)\right\},$$

where for any $\gamma \in (0,1)$, $\chi^2_{2n}(\gamma)$ is a fixed point such that $P\left(\chi^2_{2n} > \chi^2_{2n}(\gamma)\right) = \gamma$. Then, the critical region R corresponds to the

(A) most powerful test of size α for testing $H_0: \theta = 1$ against $H_1: \theta = 2$

(B) most powerful test of size $1-\alpha$ for testing $H_0^*: \theta = 2$ against $H_1^*: \theta = 1$

(C) most powerful test of size β for testing $H_0^*: \theta = 2$ against $H_1^*: \theta = 1$

(D) most powerful test of size $1-\beta$ for testing $H_0^*: \theta = 2$ against $H_1^*: \theta = 1$

Q.24 Let

$$S = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{k}\left(\frac{1}{4}\right)^k \quad \text{and} \quad T = \sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{1}{5}\right)^k.$$

Then, which of the following statements is **TRUE**?

(A) $S - T = 0$

(B) $5\,S - 4\,T = 0$

(C) $4\,S - 5\,T = 0$

(D) $16\,S - 25\,T = 0$

Q.25 Let E_1, E_2, E_3 and E_4 be four events such that

$$P(E_i|E_4) = \frac{2}{3}, i = 1,2,3;\; P\left(E_i \cap E_j^c|E_4\right) = \frac{1}{6}, i,j = 1,2,3; i \neq j \text{ and } P(E_1 \cap E_2 \cap E_3^c|E_4) = \frac{1}{6}$$

Then, $P(E_1 \cup E_2 \cup E_3|E_4)$ is equal to

(A) $\frac{1}{2}$

(B) $\frac{2}{3}$

(C) $\frac{5}{6}$

(D) $\frac{7}{12}$

Q.26 Let $a_1 = 5$ and define recursively

$$a_{n+1} = 3^{\frac{1}{4}} (a_n)^{\frac{3}{4}}, \qquad n \geq 1.$$

Then, which of the following statements is **TRUE**?

(A) $\{a_n\}$ is monotone increasing, and $\lim_{n\to\infty} a_n = 3$

(B) $\{a_n\}$ is monotone decreasing, and $\lim_{n\to\infty} a_n = 3$

(C) $\{a_n\}$ is non-monotone, and $\lim_{n\to\infty} a_n = 3$

(D) $\{a_n\}$ is decreasing, and $\lim_{n\to\infty} a_n = 0$

Q.27 Consider the problem of testing $H_0: X \sim f_0$ against $H_1: X \sim f_1$ based on a sample of size 1, where

$$f_0(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases} \text{ and } f_1(x) = \begin{cases} 2-2x, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Then, the probability of Type II error of the most powerful test of size $\alpha = 0.1$ is equal to

(A) 0.81 (B) 0.91 (C) 0.1 (D) 1

Q.28 For $a \in \mathbb{R}$, consider the system of linear equations

$$\begin{aligned} a\,x + a\,y \quad &= a+2 \\ x + a\,y + (a-1)z &= a-4 \\ a\,x + a\,y + (a-2)z &= -8 \end{aligned}$$

in the unknowns x, y and z. Then, which of the following statements is **TRUE**?

(A) The given system has a unique solution for $a = 1$

(B) The given system has infinitely many solutions for $a = 2$

(C) The given system has a unique solution for $a = -2$

(D) The given system has infinitely many solutions for $a = -2$

Q.29 Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers such that $a_n \geq 1$, for all $n \geq 1$. Then, which of the following conditions imply the divergence of $\{a_n\}_{n\geq 1}$?

(A) $\{a_n\}_{n\geq 1}$ is non-increasing

(B) $\sum_{n=1}^{\infty} b_n$ converges, where $b_1 = a_1$ and $b_n = a_{n+1} - a_n$, for all $n > 1$

(C) $\lim_{n\to\infty} \frac{a_{2n+1}}{a_{2n}} = \frac{1}{2}$

(D) $\{\sqrt{a_n}\}_{n\geq 1}$ converges

Q.30 Let E_1, E_2 and E_3 be three events such that $P(E_1) = \frac{4}{5}, P(E_2) = \frac{1}{2}$ and $P(E_3) = \frac{9}{10}$.
Then, which of the following statements is **FALSE**?

(A) $P(E_1 \cup E_2 \cup E_3) \geq \frac{9}{10}$

(B) $P(E_2 \cap E_3) \leq \frac{1}{2}$

(C) $P(E_1 \cap E_2 \cap E_3) \leq \frac{1}{6}$

(D) $P(E_1 \cup E_2) \geq \frac{4}{5}$

SECTION - B

MULTIPLE SELECT QUESTIONS (MSQ)

Q. 31 – Q. 40 carry two marks each.

Q.31 Consider the linear system $A\,\underline{x} = \underline{b}$, where A is an $m \times n$ matrix, $\underline{x}$ is an $n \times 1$ vector of unknowns and $\underline{b}$ is an $m \times 1$ vector. Further, suppose there exists an $m \times 1$ vector $\underline{c}$ such that the linear system $A\underline{x} = \underline{c}$ has **NO** solution. Then, which of the following statements is/are necessarily **TRUE**?

(A) If $m \leq n$ and $\underline{d}$ is the first column of A, then the linear system $A\underline{x} = \underline{d}$ has a unique solution

(B) If $m \geq n$, then $\text{Rank}(A) < n$

(C) $\text{Rank}(A) < m$

(D) If $m > n$, then the linear system $A\underline{x} = \underline{0}$ has a solution other than $\underline{x} = \underline{0}$

Q.32 Let A be a 3×3 real matrix such that $A \neq I_3$ and the sum of the entries in each row of A is 1. Then, which of the following statements is/are necessarily **TRUE**?

(A) $A - I_3$ is an invertible matrix

(B) The set $\{\, \underline{x} \in \mathbb{R}^3 : (A - I_3)\underline{x} = \underline{0} \,\}$ has at least two elements ($\underline{x}$ is a column vector)

(C) The characteristic polynomial, $p(\lambda)$, of $A + 2\,A^2 + A^3$ has $(\lambda - 4)$ as a factor

(D) A cannot be an orthogonal matrix

Q.33 Let $X_1, X_2, \dots, X_n$ be a random sample from $N(\theta, 1)$, where $\theta \in (-\infty, \infty)$ is unknown. Consider the problem of testing $H_0: \theta \leq 0$ against $H_1: \theta > 0$. Let $\beta(\theta)$ denote the power function of the likelihood ratio test of size α $(0 < \alpha < 1)$ for testing H_0 against H_1. Then, which of the following statements is/are **TRUE**?

(A) $\beta(\theta) > \beta(0)$, for all $\theta > 0$

(B) $\beta(\theta) < \beta(0)$, for all $\theta > 0$

(C) The critical region of the likelihood test of size α is

$$\left\{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : \sqrt{n}\,\frac{\sum_{i=1}^{n} x_i}{n} > \tau_{\alpha/2}\right\},$$

where $\tau_{\alpha/2}$ is a fixed point such that $P(Z > \tau_{\alpha/2}) = \frac{\alpha}{2}$, $Z \sim N(0,1)$.

(D) The critical region of the likelihood test of size α is

$$\left\{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : \sqrt{n}\,\frac{\sum_{i=1}^{n} x_i}{n} < \tau_{\alpha}\right\},$$

where τ_α is a fixed point such that $P(Z > \tau_\alpha) = \alpha$, $Z \sim N(0, 1)$.

Q.34 Consider the function

$$f(x, y) = 3\,x^2 + 4\,x\,y + y^2, \qquad (x, y) \in \mathbb{R}^2.$$

If $S = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$, then which of the following statements is/are **TRUE**?

(A) The maximum value of f on S is $3 + \sqrt{5}$

(B) The minimum value of f on S is $3 - \sqrt{5}$

(C) The maximum value of f on S is $2 + \sqrt{5}$

(D) The minimum value of f on S is $2 - \sqrt{5}$

Q.35 Let $f: \mathbb{R} \to \mathbb{R}$ be a twice differentiable function. Then, which of the following statements is/are necessarily **TRUE**?

(A) f'' is continuous

(B) If $f'(0) = f'(1)$, then $f''(x) = 0$ has a solution in $(0, 1)$

(C) f' is bounded on $[8,\ 10]$

(D) f'' is bounded on $(0,\ 1)$

Q.36 Let $X_1, X_2, \ldots, X_n$ $(n \geq 2)$ be independent and identically distributed random variables with probability density function

$$f(x) = \begin{cases} \frac{1}{x^2}, & x \geq 1 \\ 0, & \text{otherwise} \end{cases}$$

Then, which of the following random variables has/have finite expectation?

(A) X_1

(B) $\frac{1}{X_2}$

(C) $\sqrt{X_1}$

(D) $\min\{X_1, \ldots, X_n\}$

Q.37 A sample of size n is drawn randomly (without replacement) from an urn containing $5n^2$ balls, of which $2n^2$ are red balls and $3n^2$ are black balls. Let X_n denote the number of red balls in the selected sample. If $\ell = \lim_{n\to\infty} \frac{E(X_n)}{n}$ and $m = \lim_{n\to\infty} \frac{\mathrm{Var}(X_n)}{n}$, then which of the following statements is/are **TRUE**?

(A) $\ell + m = \frac{16}{25}$

(B) $\ell - m = \frac{3}{25}$

(C) $\ell m = \frac{14}{125}$

(D) $\frac{\ell}{m} = \frac{5}{3}$

Q.38 Let $X_1, X_2, \dots, X_n$ $(n \geq 2)$ be a random sample from a distribution with probability density function

$$f(x;\theta) = \begin{cases} \frac{1}{2\theta}, & -\theta \leq x \leq \theta \\ 0, & |x| > \theta \end{cases},$$

where $\theta \in (0, \infty)$ is unknown. If $R = \min\{X_1, X_2, \dots, X_n\}$ and $S = \max\{X_1, X_2, \dots, X_n\}$, then which of the following statements is/are **TRUE**?

(A) (R, S) is jointly sufficient for θ

(B) S is an MLE of θ

(C) $\max\{|X_1|, |X_2|, \dots, |X_n|\}$ is a complete and sufficient statistic for θ

(D) Distribution of $\frac{R}{S}$ does NOT depend on θ

Q.39 Let $X_1, X_2, \dots, X_n$ $(n \geq 2)$ be a random sample from a distribution with probability density function

$$f(x;\theta) = \begin{cases} \frac{3x^2}{\theta} e^{-x^3/\theta}, & x > 0 \\ 0, & \text{otherwise} \end{cases},$$

where $\theta \in (0, \infty)$ is unknown.

If $T = \sum_{i=1}^{n} X_i^3$, then which of the following statements is/are **TRUE**?

(A) $\frac{n-1}{T}$ is the unique uniformly minimum variance unbiased estimator of $\frac{1}{\theta}$

(B) $\frac{n}{T}$ is the unique uniformly minimum variance unbiased estimator of $\frac{1}{\theta}$

(C) $(n-1)\sum_{i=1}^{n} \frac{1}{X_i^3}$ is the unique uniformly minimum variance unbiased estimator of $\frac{1}{\theta}$

(D) $\frac{n}{T}$ is the MLE of $\frac{1}{\theta}$

Q.40 Let $X_1, X_2, \ldots, X_n$ $(n \geq 2)$ be a random sample from a distribution with probability density function

$$f(x;\theta) = \begin{cases} \theta x^{\theta-1}, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases},$$

where $\theta \in (0, \infty)$ is unknown. Then, which of the following statements is/are **TRUE**?

(A) Cramer-Rao lower bound, based on $X_1, X_2, \ldots, X_n$, for the estimand θ^3 is $9\frac{\theta^6}{n}$

(B) Cramer-Rao lower bound, based on $X_1, X_2, \ldots, X_n$, for the estimand θ^3 is $\frac{\theta^2}{n}$

(C) There does NOT exist any unbiased estimator of $\frac{1}{\theta}$ which attains the Cramer-Rao lower bound

(D) There exists an unbiased estimator of $\frac{1}{\theta}$ which attains the Cramer-Rao lower bound

SECTION – C

NUMERICAL ANSWER TYPE (NAT)

Q. 41 – Q. 50 carry one mark each.

Q.41 Let α, β and γ be the eigenvalues of $M = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 3 & 3 \\ -1 & 2 & 2 \end{bmatrix}$. If $\gamma = 1$ and $\alpha > \beta$, then the value of $2\,\alpha + 3\,\beta$ is ________________________.

Q.42 Let $M = \begin{pmatrix} 5 & -6 \\ 3 & -4 \end{pmatrix}$ be a 2×2 matrix. If $\alpha = \det(M^4 - 6I_2)$, then the value of α^2 is ____________________.

Q.43 Let $S = \{(x, y) \in \mathbb{R}^2 : 2 \leq x \leq y \leq 4\}$. Then, the value of the integral

$$\iint_S \frac{1}{4 - x}\, dx\, dy$$

is __________________________.

Q.44 Let $A = \{(x, y, z) \in \mathbb{R}^3 : 0 \leq x \leq y \leq z \leq 1\}$. Let α be the value of the integral

$$\iiint_A x\, y\, z\ dx\, dy\, dz.$$

Then, $384\,\alpha$ is equal to _____________________________.

Q.45 Let f_0 and f_1 be the probability mass functions given by

x	1	2	3	4	5	6
$f_0(x)$	0.1	0.1	0.1	0.1	0.1	0.5
$f_1(x)$	0.1	0.1	0.2	0.2	0.2	0.2

Consider the problem of testing the null hypothesis $H_0 : X \sim f_0$ against $H_1 : X \sim f_1$ based on a single sample X. If α and β, respectively, denote the size and power of the test with critical region $\{x \in \mathbb{R} : x > 3\}$, then $10(\alpha + \beta)$ is equal to __________________.

Q.46 Let $5, 10, 4, 15, 6$ be an observed random sample of size 5 from a distribution with probability density function

$$f(x;\theta) = \begin{cases} e^{-(x-\theta)}, & x \geq \theta \\ 0, & \text{otherwise} \end{cases}$$

$\theta \in (-\infty,\ 3]$ is unknown. Then, the maximum likelihood estimate of θ based on the observed sample is equal to ____________________.

Q.47 Let

$$\alpha = \lim_{n\to\infty} \sum_{m=n^2}^{2n^2} \frac{1}{\sqrt{5\,n^4 + n^3 + m}}$$

Then, $10\sqrt{5}\,\alpha$ is equal to ____________________________.

Q.48 Let X be a random variable having the probability density function

$$f(x) = \frac{1}{8\sqrt{2\pi}}\left(2\,e^{-\frac{x^2}{2}} + 3\,e^{-\frac{x^2}{8}}\right), \quad -\infty < x < \infty.$$

Then, $4\,E(X^4)$ is equal to __________________.

Q.49 Let X be a random variable with moment generating function

$$M_X(t) = \frac{1}{12} + \frac{1}{6}e^t + \frac{1}{3}e^{2t} + \frac{1}{4}e^{-t} + \frac{1}{6}e^{-2t}, t \in \mathbb{R}.$$

Then, $8\,E(X)$ is equal to ________________________.

Q.50 Let β denote the length of the curve $y = \ln(\sec x)$ from $x = 0$ to $x = \frac{\pi}{4}$.

Then, the value of $3\sqrt{2}\,(e^{\beta} - 1)$ is equal to ______________.

Q.51 – Q. 60 carry two marks each.

Q.51 Let $S \subseteq \mathbb{R}^2$ be the region bounded by the parallelogram with vertices at the points $(1,0), (3,2),$ $(3,5)$ and $(1,3)$. Then, the value of the integral $\iint_S (x + 2\,y)\,dx\,dy$ is equal to _____________.

Q.52 Let $A = \left\{(x,y) \in \mathbb{R}^2\colon\ x^2 - \frac{1}{2\sqrt{\pi}} < y < \ x^2 + \frac{1}{2\sqrt{\pi}}\right\}$ and let the joint probability density function of (X, Y) be

$$f(x,y) = \begin{cases} e^{-(x-1)^2}, & (x,y) \in A \\ 0, & \text{otherwise} \end{cases}$$

Then, the covariance between the random variables X and Y is equal to _____________.

Q.53 Let X_1 and X_2 be independent $N(0,1)$ random variables. Define

$$\mathrm{sgn}(u) = \begin{cases} -1, & \text{if } u < 0 \\ 0, & \text{if } u = 0 \\ 1, & \text{if } u > 0 \end{cases}$$

Let $Y_1 = X_1\,\mathrm{sgn}(X_2)$ and $Y_2 = X_2\,\mathrm{sgn}(X_1)$. If the correlation coefficient between Y_1 and Y_2 is α, then $\pi\alpha$ is equal to _______________________.

Q.54 Let

$$a_n = \sum_{k=2}^{n} \binom{n}{k} \frac{2^k (n-2)^{n-k}}{n^n}, \quad n = 2, 3, \dots.$$

Then, $e^2 \lim_{n\to\infty} (1 - a_n)$ is equal to _______________________.

Q.55 Let E_1, E_2, E_3 and E_4 be four independent events such that $P(E_1) = \frac{1}{2}, P(E_2) = \frac{1}{3}, P(E_3) = \frac{1}{4}$ and $P(E_4) = \frac{1}{5}$. Let p be the probability that at most two events among E_1, E_2, E_3 and E_4 occur. Then, $240\,p$ is equal to _____________.

Q.56 Let the random vector (X, Y) have the joint probability mass function

$$f(x,y) = \begin{cases} \binom{10}{x}\binom{5}{y}\left(\frac{1}{4}\right)^{x-y+5}\left(\frac{3}{4}\right)^{y-x+10}, & x = 0,1,\dots,10;\ y = 0,1,\dots,5 \\ 0, & \text{otherwise} \end{cases}$$

Let $Z = Y - X + 10$. If $\alpha = E(Z)$ and $\beta = \mathrm{Var}(Z)$, then $8\,\alpha + 48\,\beta$ is equal to ___________.

Q.57 Let $S = \left\{(x, y) \in \mathbb{R}^2 : 0 \le x \le \pi,\ \min\{\sin x, \cos x\} \le y \le \max\{\sin x, \cos x\}\right\}$.

If α is the area of S, then the value of $2\sqrt{2}\,\alpha$ is equal to ________________.

Q.58 The number of real roots of the polynomial

$$f(x) = x^{11} - 13\,x + 5$$

is ______________.

Q.59 Let $\alpha = \lim_{n\to\infty}\left(1 + n\sin\frac{3}{n^2}\right)^{2n}$. Then, $\ln\alpha$ is equal to ________________

Q.60 Let $\phi : (-1,\ 1) \to \mathbb{R}$ be defined by

$$\phi(x) = \int_{x^7}^{x^4} \frac{1}{1+t^3}\,dt.$$

If $\alpha = \lim_{x\to 0}\frac{\phi(x)}{e^{2x^4}-1}$, then $42\,\alpha$ is equal to ________________

END OF THE QUESTION PAPER